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The Mass of the Strange Quark

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Abstract

Sum rules in QCD are discussed and finite energy and Laplace transform sum rules are used to determine the strange-quark mass. We use improved QCD input and experimental data to obtain for the invariant mass:

$$\hat{m}_s = 266 \pm 29 \ MeV,$$

or for the running mass at 1 GeV:

$$\overline{m}_s(1 \ GeV) = 194 \pm 4 \ MeV.$$

Contents

Chapter 1

Introduction

In the last two decades, quarks have risen in status from being a mere speculation to explain hadronic classifications, to extremely well-documented particles known even to most non-scientists. Concurrent with this rise in fame has come the neccessity of accurately knowing their properties. In this thesis, we deal with the mass of the strange quark.

When talking about quark masses, we must distinguish between the so-called constituent quark masses, and the Lagrangian quark masses (also known, for historical reasons, as the current algebra quark masses). [1]

Constituent quark masses arise from the confinement of quarks to within a small volume. We know from the uncertainty principle that if a particle is confined to, say, 1 fermi, then $\Delta p \sim 1/\Delta x \sim 200\ MeV$. So such a particle is given a mass of $\sim$ 200 MeV by its being confined. Involving as it does both short distance *and* non-perturbative interactions, the constituent masses cannot be calculated well. However, this does explain how a proton with mass 938 MeV could be made up of three quarks, each with a current algebra mass of only a few MeV but a constituent mass of a few hundred MeV; in fact, massive hadrons could in principle be constructed from massless quarks.

Constituent masses are distinct from the current algebra masses, the masses appearing in the Lagrangian. Current algebra masses are an inherent property of the quarks, and add to the masses obtained from confining the quarks to a small volume.

The strange quark mass entering the Lagrangian is important, because it measures the size of chiral $SU(3) \otimes SU(3)$ symmetry breaking, as well as flavour $SU(3)$ symmetry breaking. Knowledge of m_s is also important to weak hadronic physics involving strangeness.

From current algebra, we obtain $\hat{m}_s = 288 \pm 48\ MeV$, while in reference [2], the running mass has been calculated as $\overline{m}_s(1\ GeV) = 192 \pm 15\ MeV$. ($\hat{m}_s$ and $\overline{m}_s$ are discussed in chapter 2.)

Recently, theoretical work done by the authors of [3], and improved phenomenological data [4], has given rise to the situation where the above estimates can be improved. This is what we attempt.

In chapter 2, we discuss QCD. We deal first with a derivation of the QCD Lagrangian and its quantization, and then discuss renormalization, leading up to a discussion of the running mass $\overline{m}_s$ and the invariant mass $\hat{m}_s$.

In chapter 3, we give an overview of various types of sum rules in QCD, in particular, the Laplace, Gaussian and finite energy sum rules.

In chapter 4 we obtain the sum rules specific to the strange quark mass, and discuss the phenomenological input.

In chapter 5, we give a description of the method of calculation, and give the results of the calculations.

Chapter 2

Quantum Chromodynamics

2.1 The motivation for quarks and colour

In the 50's and 60's, a large number of hadrons were discovered. Many were very short lived, and appeared not to be fundamental. It was found that all hadrons could be arranged systematically in multiplets of fixed spin and parity, and these multiplets were higher dimensional representations of SU(3) which could be constructed from the fundamental triplet and anti-triplet representations of SU(3). [5]

In 1964 Gell-Mann and Zweig put forward the quark hypothesis, which proposed that hadrons were built up from particles which transformed as the fundamental representation of SU(3). These particles were called quarks, and came in three flavours: up, down and strange. Baryons were supposed to be made of three quarks, and mesons of a quark-antiquark pair. Later two new quarks, charm and beauty, were introduced to explain new resonances which had been observed. Originally, the quark model was assumed to be just a convenient explanation, arising from group theory, of the above-mentioned hadronic multiplets. Howevever, evidence from deep inelastic scattering of electrons off protons implied that the electrons were scattering from pointlike particles, called partons, *within* the proton, and that these partons carried about half the proton's momentum [6]. Furthermore, the quark model was extremely successful in explaining the structure of hadrons, and even predicted the existence of a particle like the Ω^- three years before its discovery. There is now no doubt that quarks do exist, and although only five have been discovered, a sixth, the top quark, is believed to exist.

Problems from quark statistics, and also from disparities between predicted and observed cross sections and decays in certain experiments, led to the hypothesis of quarks having an additional degree of freedom. Each quark of a certain flavour is assumed to have an additional quantum number, that of colour. There are three possible colours; red, blue or green. Although quarks exist in one of these three states, all hadrons are assumed to be colour singlets.

These three colour states are then associated with the fundamental representation $\underline{3}$ of $SU(3)_{colour}$. The theory of quantum chromodynamics is based on the above assumptions, and is obtained by constructing a Lagrangian which is invariant under local gauge transformations of the group $SU(3)_{colour}$.[7]

2.2 The QCD Lagrangian and its quantization

The QCD Lagrangian may be derived as follows: We start with the Lagrangian for free quark fields with mass m

$$\mathcal{L}_0(x) = i\overline{\psi}^{\alpha}(x)\gamma^{\mu}\partial_{\mu}\psi_{\alpha}(x) - m\overline{\psi}^{\alpha}(x)\psi_{\alpha}(x) \tag{2.1}$$

where $\psi_{\alpha}(x)$ are the quark fields, the index $\alpha = 1,2,3$ refers to colour and the flavour index has been dropped for convenience.

This Lagrangian is invariant under global gauge transformations, but not under local gauge transformations of the type

$$\psi_{\alpha}(x) \to \psi'_{\alpha}(x) = G(x)\psi_{\alpha}(x) \equiv \exp\left(-ig\vec{T}.\vec{\theta}(x)\right)\psi_{\alpha}(x)$$

where g is the QCD coupling constant, $\theta^{(a)}(x)$ $(a = 1,2,\ldots,8)$ are eight arbitrary real space-time functions, and $T^{(a)}$ are the eight generators of SU(3). The generators $T^{(a)}$ satisfy the algebra $[T^{(a)}, T^{(b)}] = if_{abc}T^{(c)}$, where f_{abc} are the structure constants of SU(3). In the fundamental representation we may write the generators as $\left(T^{(a)}\right)_{\alpha\beta} = \frac{1}{2}\lambda^{(a)}_{\alpha\beta}$, where the $\lambda^{(a)}_{\alpha\beta}$ are the Gell-Mann matrices, and in the adjoint representation we may write $\left(T^{(a)}\right)_{bc} = -if_{abc}$.

In order to make the Lagrangian invariant under such transformations, we must substitute for the derivative ∂_{μ} the covariant derivative

$$D_{\mu} = \partial_{\mu} - ig\vec{T}.\vec{W}_{\mu}(x) \equiv \partial_{\mu} + gW_{\mu}(x) \quad \text{with} \quad W_{\mu}(x) \equiv \frac{1}{i}\vec{T}.\vec{W}_{\mu}(x) \tag{2.2}$$

where we have introduced the eight gauge fields $W^{(a)}_{\mu}$. These fields represent the gluons.

For the Lagrangian to be invariant under the given transformation, $\psi_{\alpha}(x)$ and $D_{\mu}\psi_{\alpha}(x)$ must transform alike under $G(x)$. This means that $W_{\mu}(x)$ must transform as

$$W_{\mu}(x) \to W'_{\mu}(x) = G(x)W_{\mu}(x)G^{-1}(x) - \frac{1}{g}\left(\partial_{\mu}G(x)\right)G^{-1}(x).$$

So, merely by the fact that we wanted a locally gauge invariant theory, we have been forced to introduce the gluons and a coupling between the quarks and gluons.

To fix the eqations of motion of the gluon fields, we add to the Lagrangian the term

$$-\frac{1}{4}F^{(a)}_{\mu\nu}(x)F^{\mu\nu}_{(a)}(x) \tag{2.3}$$

where $F_{\mu\nu}(x) = -ig\left(\partial_{\mu}\vec{W}_{\nu}(x) - \partial_{\nu}\vec{W}_{\mu}(x)\right).\vec{T} - ig^2 f_{abc}T^{(c)}W^{(a)}_{\mu}W^{(b)}_{\nu}$.

Then, including flavour indices, we may write the QCD Lagrangian (before quantization) as

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^{(a)}_{\mu\nu}(x)F^{\mu\nu}_{(a)}(x) + i\sum_{j=1}^{N_f}\overline{\psi}^{\alpha}_j(x)\gamma^{\mu}\left(D_{\mu}\right)_{\alpha\beta}\psi^{\beta}_j(x) - \sum_{j=1}^{N_f}m_j\overline{\psi}^{\alpha}_j(x)\psi_{\alpha j}(x). \tag{2.4}$$

We now consider some global symmetries of the Lagrangian. $\mathcal{L}_{QCD}$ is invariant under the U(1) transformation

$$\psi(x) \to \psi'(x) = \exp\left(-i\theta\mathrm{I}\right)\psi(x)$$

(I is the unit matrix) which implies the conservation of baryon number.

$\mathcal{L}_{QCD}$ also has a $\mathrm{U_1(1)} \otimes \mathrm{U_2(1)} \otimes \ldots \otimes \mathrm{U_{N_f}(1)}$ global symmetry; i.e. it is invariant under the transformations

$$\psi_j^\alpha(x) \to \psi_j'^\alpha(x) = \exp(-i\theta_j)\,\psi_j^\alpha(x) \tag{2.5}$$

acting on each of the flavours j ($j = 1, 2, \ldots, N_f$).

If the masses m_j corresponding to each flavour are equal then $\mathcal{L}_{QCD}$ is invariant under $\mathrm{SU}(N_f)$, where N_f is the number of flavours present in the regime one is dealing with.

Since the masses of the up and down quarks (each only a few MeV) are very small compared to the hadronic scale (about 1 GeV), SU(2) (isospin invariance) is an almost exact symmetry of $\mathcal{L}_{QCD}$. To know the magnitude of SU(3) symmetry breaking, one needs to know the mass of the strange quark.

Finally, if the masses of all the flavours one is dealing with are set equal, $\mathcal{L}_{QCD}$ is invariant under the transformation

$$\psi^\alpha(x) \to \psi'^\alpha(x) = \exp\left(-i\theta^{(A)}T^{(A)}\right)\psi^\alpha(x) \tag{2.6}$$

and if the masses are set to zero, $\mathcal{L}_{QCD}$ is invariant under

$$\psi^\alpha(x) \to \psi'^\alpha(x) = \exp\left(-i\theta_5^{(A)}T^{(A)}\gamma_5\right)\psi^\alpha(x). \tag{2.7}$$

$\theta^{(A)}$ and $\theta_5^{(A)}$ are constant parameters and $T^{(A)}$ are the generators of $\mathrm{SU}(N_f)$ acting on the quark flavour components. The conserved Noether currents are

$$V_\mu^{(A)}(x) \equiv \overline{\psi}^i(x)\gamma_\mu T_{ij}^{(A)}\psi^j(x) \tag{2.8}$$

and

$$A_\mu^{(A)}(x) \equiv \overline{\psi}^i(x)\gamma_\mu\gamma_5 T_{ij}^{(A)}\psi^j(x) \tag{2.9}$$

with associated charges

$$Q^{(A)}(t) \equiv \int d^3x V_0^{(A)}(\vec{x}, t) \tag{2.10}$$

and

$$Q_5^{(A)}(t) \equiv \int d^3x A_0^{(A)}(\vec{x}, t).$$

eeqn

Then the charges

$$Q_L^{(A)} = Q^{(A)} - Q_5^{(A)}$$

and

$$Q_R^{(A)} = Q^{(A)} + Q_5^{(A)} \tag{2.12}$$

are the generators of chiral $\mathrm{SU_L(N_f)} \otimes \mathrm{SU_R(N_f)}$, which for three flavours is chiral $\mathrm{SU_L(3)} \otimes \mathrm{SU_R(3)}$.

So knowledge of how much the strange quark mass differs from zero also enables one to determine the magnitude of chiral SU(3) $\otimes$ SU(3) symmetry breaking.

So, on to quantization.

The Lagrangian obtained above cannot be covariantly quantized, since the canonical conjugate momentum

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 W_0(x))} = 0. \tag{2.13}$$

Using this, it can be seen from the equal time commutation relations that π^0 commutes with all other operators, unlike the π^i, and hence the theory is not covariant. This is remedied by adding to the Lagrangian the gauge fixing term

$$-\frac{1}{2a}\partial_\mu \vec{W}^\mu(x).\partial_\nu \vec{W}^\nu(x) \tag{2.14}$$

This is like the procedure in QED, where a similar term had to be added. Unfortuneately, the addition of this term in QCD introduces non-physical longitudinal and timelike gluons into the theory, and unlike in QED, the probabilities of observing these non-physical particles do not cancel. Since the theory must be unitary in the physical sector, in other words not allow transitions to non-physical particles, we are forced to add another term, the so called Faddeev-Popov term . This may be written

$$-\partial_\mu \overline{\varphi}_{(a)}(x)\left[\delta_{ab}\partial^\mu - ig\left(-if_{cab}\right)W^\mu_{(c)}(x)\right]\varphi_{(b)}(x) \tag{2.15}$$

where $\varphi_{(a)}$ are the eight ghost fields.

The probability of observing these ghosts exactly cancels the probability of observing the above-mentioned non-physical gluons, and so the theory is once again acceptable.

One may now write the full Lagrangian

$$\begin{aligned}
\mathcal{L}_{QCD}(x) \;=\; & -\frac{1}{4}F^{(a)}_{\mu\nu}(x)F^{\mu\nu}_{(a)}(x) - \frac{1}{2a}\partial_\mu \vec{W}^\mu(x).\partial_\nu \vec{W}^\nu(x) \\
& - \partial_\mu \overline{\varphi}_{(a)}(x)\left[\delta_{ab}\partial^\mu - ig\left(-if_{cab}\right)W^\mu_{(c)}(x)\right]\varphi_{(b)}(x) \\
& + i\sum_{j=1}^{N_f}\overline{\psi}^\alpha_j(x)\gamma^\mu\left[\delta_{\alpha\beta}\partial_\mu - ig\sum_{a=1}^{8}\frac{1}{2}\lambda^{(a)}_{\alpha\beta}W^{(a)}_\mu(x)\right]\psi^\beta_j(x) \\
& - \sum_{j=1}^{N_f} m_j\overline{\psi}^\alpha_j(x)\psi_{\alpha j}(x).
\end{aligned} \tag{2.16}$$

The gauge fixing term added to the Lagrangian above has broken the gauge invariance, but there is a generalization of gauge invariance, the so-called Becchi-Rouet-Stora or B-R-S invariance, applicable to the quantum theory. The B-R-S transformations are

$$\begin{aligned}
W^{(a)}_\mu(x) &\rightarrow W^{(a)\prime}_\mu(x) = W^{(a)}_\mu(x) + \omega\left(D_\mu\varphi\right)^{(a)}(x) \\
\psi(x) &\rightarrow \psi'(x) = \exp\left(-ig\omega\vec{T}.\vec{\varphi}(x)\right)\psi(x) \\
\overline{\varphi}^{(a)}(x) &\rightarrow \overline{\varphi}'^{(a)}(x) = \overline{\varphi}^{(a)}(x) + \frac{\omega}{a}\partial^\mu W^{(a)}_\mu(x) \\
\varphi^{(a)}(x) &\rightarrow \varphi'^{(a)}(x) = \varphi^{(a)}(x) - \frac{1}{2}g\omega\varphi^{(b)}(x)f_{abc}\varphi^{(c)}(x).
\end{aligned} \tag{2.17}$$

All the relations among Green's functions which result from the local gauge invariance of the theory, the Slavnov-Taylor identities, are generated by the B-R-S transformations.

From the Lagrangian we have obtained, we may derive the equations of motion, the non-zero canonical commutation rules for the physical fields, and the free propagators for the quark, gluon and ghost fields. Finally, one may obtain the Feynman rules for QCD (see refs [8], [9]).

2.3 Renormalization

Calculations in QCD of self-energy and vertex diagrams are straightforward at the tree level, but as soon as one or more loops are introduced, the diagrams become ultra-violet divergent and hence meaningless. The procedure of ridding the theory of these infinities is called renormalization.

The first step is to regularize the theory, using the so-called dimensional regularization [10],[11],[12]. By reducing the dimension of space-time from $D = 4$ to $D = n = 4 - \varepsilon$, the divergent integrals may be made meaningful if n is small enough. Accompanying this, one needs to redefine the Dirac algebra for $D = 4 - \varepsilon$ dimensions, and change the dimensions of quantities such as the fields, coupling constants, masses and gauge parameters appearing in the Lagrangian.

This having been done, it is easy to identify the divergent parts of the Lagrangian. These are removed by adding appropriate counter terms to the Lagrangian. These counterterms exactly cancel the divergent parts of the theory. So

$$\mathcal{L}_{QCD} \rightarrow \mathcal{L}_{QCD} + \Delta\mathcal{L}_{QCD} \tag{2.18}$$

There are terms corresponding to each term in the Lagrangian, each with a different Δ_i. It has been shown that with each of the Δ_i's taken as a power series in α_s, the counterterms are enough to remove divergences to all orders. Here

$$\alpha_s = g^2/4\pi(\nu^2)^{(-\varepsilon/2)} \tag{2.19}$$

where ν is an arbitrary mass scale arising from the regularization procedure. This is the same as saying that the theory is renormalizable.

If we write $Z_i = 1 - \Delta_i$, for each of the counterterms Δ_i, we can rescale the fields, coupling constant, mass and gauge parameter in terms of the different Z_i.

We may then write

$$\begin{aligned} \mathcal{L}_{0QCD} &= \mathcal{L}_{QCD} + \Delta\mathcal{L}_{QCD} \\ &\quad -\frac{1}{4}\vec{F}^{0}_{\mu\nu}(x).\vec{F}^{\mu\nu}_{0}(x) - \frac{1}{2a_0}\left(\partial_\mu \vec{W}^{\mu}_{0}\right)^2 \\ &\quad + \frac{i}{2}\sum_{j=0}^{N_f}\overline{\psi}_{j0}\gamma^\mu D^0_\mu \psi_{j0} - \sum_{j=0}^{N_f} m_{j0}\overline{\psi}_{j0}\psi_{j0} \\ &\quad - \partial_\mu \vec{\overline{\varphi}}_0 . D^\mu_0 \vec{\varphi}^0 \end{aligned} \tag{2.20}$$

provided certain constraints between the Z_i are obeyed. This is the bare Lagrangian, and the quantities on the right hand side are the bare fields, coupling constant, masses and gauge parameter.

There are a number of different schemes which may be used to eliminate the singularities in the theory.

In the minimal subtraction, or MS scheme , the renormalization constants Z_i are chosen to cancel the poles in $1/\varepsilon$ in the propagators.

In the modified minimal subtraction scheme $\left(\overline{\mathrm{MS}}\right)$, the Z_i are chosen to eliminate terms containing $(1/\varepsilon - \ln 4\pi + \gamma)$ in the propagators. There are also other renormalization schemes, the μ -renormalization scheme and Weinberg's W-scheme, which involve subtracting from the Green's functions their value at some Euclidean point. The work in this thesis is performed in the $\overline{\mathrm{MS}}$ scheme.

If we have a renormalized Green's function Γ and a bare Green's function Γ_0, they are related by the equation

$$\Gamma(p_1,\ldots,p_N;\alpha_s,a,m_i,\mu) = Z_\Gamma(\mu,\varepsilon)\Gamma_0(p_1,\ldots,p_N;\alpha_s^0,a0,m_i^0,\varepsilon). \tag{2.21}$$

The fundamental equation of the renormalization group can then be derived :

$$\begin{aligned} &\left\{-\frac{\partial}{\partial t} + \beta(\alpha_s)\alpha_s\frac{\partial}{\partial\alpha_s} + \beta_a(\alpha_s)\frac{\partial}{\partial a}\right. \\ &\left. - \sum_i [1+\gamma_i(\alpha_s)]\, x_i\frac{\partial}{\partial x_i} + D - \gamma_\Gamma(\alpha_s)\right\} \\ &\bullet\ \Gamma(e^t p_1,\ldots,e^t p_N;\alpha_s,a,x_i,\mu) \qquad = \quad 0 \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} x_i = m_i/\mu \quad &, \quad \alpha_s\beta(\alpha_s) = \mu\frac{d\alpha_s}{d\mu}, \\ -\gamma_i(\alpha_s) = \frac{\mu}{m_i}\frac{dm_i}{d\mu} \quad &, \quad \beta_a(\alpha_s = \mu\frac{da}{d\mu}, \end{aligned} \tag{2.23}$$

and D is the dimension of Γ in mass units. The expression γ_Γ is a function of the renormalization constants Z_i and the number of external quark, gluon and ghost lines.

The solution to the fundamental equation may be found (see [8],[9]) by first solving the differential equations for the effective, or running , coupling constant, mass, and gauge parameter:

$$\begin{aligned} \frac{d\overline{\alpha}_s(t,\alpha_s)}{dt} &= \overline{\alpha}_s(\beta,\alpha_s) \quad \text{with } \overline{\alpha}_s(0,\alpha_s) = \alpha_s \\ \frac{d\overline{x}_i(t,\alpha_s)}{dt} &= -[1+\gamma_i(\alpha_s)]\,\overline{x}_i \quad \text{with } \overline{x}_i(0,\alpha_s) = x_i \\ \frac{d\overline{a}(t,\alpha_s)}{dt} &= \beta_a(\alpha_s) \quad \text{with } \overline{a}(0,\alpha_s) = a \end{aligned} \tag{2.24}$$

Then the general solution to the fundamental equation is

$$\Gamma(e^t p_1,\ldots,e^t p_N;\alpha_s,a,x_i,\mu) = \Gamma(p_1,\ldots,p_N;\overline{\alpha_s},\overline{a},\overline{x_i},\mu)\exp\left\{ tD - \int_0^t dt'\gamma_\Gamma\left[\overline{\alpha}_s(t',\alpha_s\right]\right\} \tag{2.25}$$

Solving the equation

$$\frac{d\overline{\alpha}_s(t,\alpha_s)}{dt} = \overline{\alpha}_s(\beta,\alpha_s) \tag{2.26}$$

to two loops, we get the result

$$\overline{\alpha}_s\left[-q^2\right]=\overline{\alpha}^{(2)}\left(\frac{q^2}{\Lambda^2}\right)\left\{1-\frac{\beta_2}{\beta_1}\frac{1}{\pi}\overline{\alpha}^{(2)}\left(\frac{q^2}{\Lambda^2}\right)\ln\left(\frac{-q^2}{\Lambda^2}\right)\right\} \tag{2.27}$$

where

$$\overline{\alpha}^{(2)}\left(\frac{q^2}{\Lambda^2}\right)\equiv\pi/\left(\frac{-\beta_1}{2}\ln\left(\frac{-q^2}{\Lambda^2}\right)\right) \tag{2.28}$$

and for three flavours $\beta_1=-9/2,\beta_2=-8$.

Λ is a parameter that must be determined experimentally.

Similarly, solving for the effective mass to two loops

$$\begin{aligned}\overline{m}_i\left(-q^2\right) &= \hat{m}_i/\left(\frac{1}{2}\left(\ln\frac{-q^2}{\Lambda^2}\right)\right)^{-\gamma_1/\beta_1}\\ &\bullet\left\{1+\frac{\gamma_2-\gamma_1\beta_2/\beta_1}{\beta_1^2}\frac{1}{\frac{1}{2}\ln\left(-q^2/\Lambda^2\right)}-\frac{\gamma_1\beta_2\ln\ln\left(-q^2/\Lambda^2\right)}{\beta_1^3\frac{1}{2}\ln\left(-q^2/\Lambda^2\right)}\right\}\end{aligned} \tag{2.29}$$

with $\gamma_1=2,\gamma_2=7.5833$. The constants $\hat{m}_i$, the so-called invariant masses, must be determined in order to be able to determine the running (or effective) mass $\overline{m}_i$. In this work we calculate $\hat{m}_s$ from QCD sum rules and hence obtain $\overline{m}_s$.

Chapter 3

Sum rules in QCD

Ideally, the ultimate theory of the strong interaction should be solvable for the complete range of momenta from zero to infinity. However, the best theory that we have so far for the strong interaction is QCD and it falls short of this ideal.

QCD works well for short distances, the so-called hard processes. As the momentum is increased, the effective coupling constant becomes small (that is, the theory becomes assymptotically free) and perturbation theory may be applied.

On the other hand, we would like to be able to work out quantities in QCD for small momenta (large distances), and , for example, evaluate the hadron spectrum or explain confinement. We cannot easily do this, because for large momentum $\overline{\alpha}_s$ becomes large and perturbation theory breaks down.

So in an attempt to solve QCD in the non-perturbative region, other methods have been used, both "brute force" and analytical. Of the brute force methods, lattice gauge theories have offered some hope but these involve a lot of computer time. Of the analytical attempts at solving long range QCD, the most promising method is the sum rule approach.[8]

The sum rule approach was first tried by Shifman, Vainshtein and Zakharov in their paper of 1979 [13]. Instead of trying to explain confinement, it assumes it's existence, and parametrizes our ignorance of long distance behavior by means of the so-called condensates.

The general technique may be described as follows: One begins with the consideration of some two point function involving, for example , a conserved current

$$\begin{aligned} \Pi_{\mu\nu}(q) &= i\int d^4x e^{iqx} < 0|\, T\left(J_\mu(x) J_\nu^\dagger(0) \right)\, |0> \\ &= (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2) \end{aligned} \tag{3.1}$$

Now, Wilson proposed [14] that one may make an Operator Product Expansion (O.P.E) for the T product of two currents

$$A(x)B(0) \cong \sum_n C_n(x) O_n \quad \text{(provided } x^\mu \to 0) \tag{3.2}$$

where O_n are local spin zero operators with dimension n. There are spin zero operators with d = 0 (the unit operator), as well as d = 4, 6, 8,...

Although the above expansion was derived for perturbation theory, we assume that it holds too in regions where the theory is no longer purely perturbative. This is true provided that only the first few non-perturbative terms are taken. Also the above expansion holds only in the weak sense, meaning that we must sandwich it between initial and final states. In perturbation theory only the first term would survive, but non-perturbative effects cause non-vanishing expectation values of higher order operators.

We then have

$$\begin{aligned} \Pi(q^2) &= \sum_n C_n(q) < 0|\, O_n \,|0 > \\ &= \left[\ (\text{perturbationtheory}) + \sum_{k=2} Z_k (\frac{1}{Q^2})^k \ \right] \end{aligned} \tag{3.3}$$

The non-perturbative terms, then, turn out to be a power series in $1/(Q^2)$. The Z_k involve the condensates. The only condensates that we have dealt with in this thesis are those corresponding to d = 4 (k = 2). They are $< \alpha_s G^2 >$, the gluon condensate, and $< m_u \bar{u} u - m_s \bar{s} s >$, the quark condensate. To obtain numerical values for the condensates, we must look at sum rules where all the parameters (except the condensates) are known or can be estimated, and solve for the condensates.

So, we now have an expression for $\Pi(Q^2)$ which we can calculate in QCD.

On the other hand, from Cauchy's theorem we may obtain the dispersion relation

$$\Pi(Q^2) = \frac{1}{\pi} \int \frac{Im\Pi(s) ds}{s + Q^2}, \quad \text{where} \quad Q^2 = -q^2 \tag{3.4}$$

This is valid up to some number of subtractions, which are necessary to render $\Pi(Q^2)$ finite. The number depends on the choice of currents in the two point function. If we choose to work with the divergences of vector currents, then, as can be seen from the assymptotic freedom behaviour calculable in QCD, as $s \rightarrow \infty$, $1/\pi \; Im\Pi(s)$ increases like s. So in this case two subtractions are neccessary to get rid of the infinities. We may dispense with these subtractions by taking the second derivative with respect to Q^2, in which case the weight function $1/(s + Q^2)$ becomes $1/(s + Q^2)^3$. Thus

$$\Pi''(Q^2) = \frac{d^2 \Pi(Q^2)}{dQ^2} = \frac{1}{\pi} \int \frac{Im\Pi(s) ds}{(s + Q^2)^3} \tag{3.5}$$

We may then insert the hadronic spectral function $1/\pi Im\Pi(s)$, which we know from experment up to some threshold value s_0. We add to this a term to account for the hadronic continuum for values of s above s_0.

Hence we have the spectral function

$$\frac{1}{\pi} Im\Pi(s)|_{HAD} = \frac{1}{\pi} Im\Pi(s)|_{RES}\, \theta(s_0 - s) + \frac{1}{\pi} Im\Pi(s)|_{AF}\, \theta(s - s_0) \tag{3.6}$$

where the term $1/\pi \; Im\Pi(s)|_{RES}$ is the resonance contribution, obtained from experiment; and $1/\pi \; Im\Pi(s)|_{AF}$ is the assymptotic freedom expression obtained from QCD.

We then equate the expression for the second derivative of $\Pi(Q^2)$ which we obtain from QCD with the second derivative of $\Pi(Q^2)$ we obtain from the hadronic spectral function, to get the sum rule

$$\Pi''(Q^2)|_{QCD} = \frac{1}{\pi} \int \frac{Im\Pi(s)|_{HAD} \; ds}{(s + Q^2)^3} \tag{3.7}$$

We may transform this to various other sum rules by applying different operators to it. Then the weight function $1/(s+Q^2)^3$ will be transformed to a different weight function.

Ideally, we would like to obtain a sum rule which emphasizes the low energy contribution to the integral on the right hand side, since the experimental input we have is from the resonance, or low energy region. Furthermore, we would like a sum rule which supresses higher orders in $1/Q^2$ on the left hand side (since we can take only the first few terms in $1/Q^2$ anyway). These two requirements are in general self-contradictory, but there is a transformation which does achieve both of these improvements to the sum rule.

The Laplace transform is defined by

$$\hat{L} = \lim_{Q^2\to\infty, N\to\infty, Q^2/N=M^2} \frac{(-1)^N}{(N-1)!}\left(Q^2\right)^N \frac{d^N}{(dQ^2)^N} \tag{3.8}$$

The factor $1/(s+Q^2)^3$ transforms under $\hat{L}$ to $1/M^6\ \exp(-s/M^2)$, which is small for large s; and on the RHS the terms $1/(Q^2)^k$ transform under $\hat{L}$ to $1/k!(1/Q^2)^k$.

So the Laplace transform increases the importance of the low energy contribution to the integral, by exponentially cutting off the high energy contribution. It simultaneously improves the approximation to the series in $1/Q^2$ by the first few terms, by factorially suppressing higher order terms.

The above are known as Laplace transform sum rules. By applying different transformations, or series of transformations, we may obtain other types of sum rules as might be appropriate to the particular calculation we wish to perform.

Gaussian sum rules are obtained by applying the Gauss-Weierstrass transform, which is the following:

We first apply the operator

$$\hat{L} = \lim_{Q^2\to\infty, N\to\infty, N/Q^2\equiv\sigma} \frac{(-1)^N}{(N-1)!}\left(Q^2\right)^N \frac{d^N}{(dQ^2)^N} \tag{3.9}$$

and then the operator

$$\hat{L} = \lim_{\sigma^2\to\infty, N\to\infty, N/\sigma^2\equiv\tau} \frac{(-1)^N}{(N-1)!}\left(\sigma^2\right)^N \frac{d^N}{(d\sigma^2)^N} \tag{3.10}$$

and then divide by 2τ to obtain the transform.

The Gauss-Weierstrass transform of the spectral function turns out to be

$$G(\hat{s},\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_0^\infty ds \exp\left[-\frac{(s-\hat{s})^2}{4\tau}\right]\frac{1}{\pi}Im\Pi(s), \tag{3.11}$$

which is a convolution of the spectral function with a Gaussian centred at arbitrary $\hat{s}$ with width $\sqrt{2\tau}$.

We may make an analogy between the Gaussian sum rules and the heat equation by noting that $G(s,\tau)$ obeys the partial differential equation of the heat equation

$$\frac{\partial^2 G(s,\tau)}{(\partial s)^2} = \frac{\partial G(s,\tau)}{\partial \tau} \tag{3.12}$$

The variable s is analogous to position and τ to time, and $1/\pi\ Im\Pi(s)$ is analogous to the initial heat distribution in a rod. $G(s,\tau)$ is equivalent to the evolution of the heat distribution in the rod.

We are interested in the so-called finite energy sum rules (FESR). These may be derived from the Gaussian sum rules by appealing to the conservation of "heat", and the assymptotic freedom property of QCD. We choose some value s_0 above which we assume that the hadronic spectral function is given by the QCD assymptotic behavior, and split the integral over the hadronic spectral function into an integral up to s_0 and an integral from s_0 to infinity. Some manipulation then leads to the finite energy sum rules (see [15],[16]).

Both FESR and Laplace transform sum rules will be given in the next chapter together with the appropriate two point functions for obtaining the strange quark mass.

Chapter 4

Sum rules for the strange quark mass

To obtain the specific sum rules for the strange quark mass, we begin by defining the two point functions [3]

$$\Pi_{\mu\nu}(q^2) = i \int d^4x e^{iqx} < 0|T\left(V_\mu(x)V_\nu^\dagger(0)\right)|0 >, \tag{4.1}$$

the vector current correlator, and

$$\Pi_S(q^2) \equiv \psi(q^2) = i \int d^4x e^{iqx} < 0|T\left(\partial^\mu V_\mu(x)\partial^\nu V_\nu^\dagger(0)\right)|0 >, \tag{4.2}$$

the scalar correlator. In the above, $V_\mu(x) = :\overline{s}(x)\gamma_\mu u(x):$ is the strangeness changing vector current.

Then there is a Ward identity which relates the above two correlators:

$$q^\mu q^\nu \Pi_{\mu\nu}(q^2) = \Pi_S(q^2) - \Pi_S(0) \tag{4.3}$$

and also enables us to decompose the vector current correlator as

$$\Pi_{\mu\nu}(q^2) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi_V(q^2) + \frac{q_\mu q_\nu}{q^4}\left(\Pi_S(q^2) - \Pi_S(0)\right) \tag{4.4}$$

Then let us consider the function

$$\xi(q^2) = \frac{\partial}{\partial q^2}\left(\frac{\Pi_S(q^2)}{q^2}\right) \tag{4.5}$$

which is defined up to the subtraction constant $\psi(0) = - < m_s(\overline{s}s - \overline{u}u) >$.

$\xi(Q^2)$ (where $Q^2 = -q^2$) satisfies the dispersion relation

$$\xi(Q^2) = \frac{1}{\pi}\int_0^\infty \frac{ds}{s}\frac{Im\psi(s)}{(s+Q^2)^2}. \tag{4.6}$$

In reference [3], $\Pi_S(q^2)$ is calculated to two loops, using the renormalization group to sum divergent series of leading mass singular logarithms. The result is (where we have taken

$N_f = 3$ and have assumed that the mass of the up quark is negligible compared to the strange quark):

$$\Pi_S(q^2) = \frac{3}{8\pi^2}\left\{q^2 L_{0,0} + L_{2,0} + q^{-2}(L_4^{gc} + L_4^{qc} + L_{4,0}^{pt}) + O(Q^{-4})\right\} \tag{4.7}$$

where

$$\begin{aligned}
L_{0,0} &= const - 4\overline{m}_s^2\left(\frac{\pi}{\overline{\alpha}_s} - \frac{5}{2} + O(\overline{\alpha}_s)\right), \\
L_{2,0} &= const - \frac{24}{21}\overline{m}_s^4\left(\frac{\pi}{\overline{\alpha}_s} - \frac{11}{24} + O(\overline{\alpha}_s)\right), \\
L_4^{qc} &= -\frac{4\pi^2}{3}\overline{m}_s^2 < m_s\overline{s}s > \left[1 + \frac{11}{3}\frac{\overline{\alpha}_s}{\pi} + 2\frac{< \overline{u}u >}{< \overline{s}s >}\left\{1 + \frac{14}{3}\frac{\overline{\alpha}_s}{\pi}\right\} + O(\alpha_s^2)\right], \\
L_{4,0}^{pt} &= \frac{12}{21}\overline{m}_s^6\left(\frac{\pi}{\overline{\alpha}_s} - \frac{7}{12} + O(\overline{\alpha}_s)\right), \\
L_4^{gc} &= \frac{\pi}{3}\overline{m}_s^2 < \alpha_s G_{\mu\nu}^a G_a^{\mu\nu} > (1 + O(\alpha_s)\ ).
\end{aligned} \tag{4.8}$$

We get from this

$$\begin{aligned}
\xi(Q^2) &= \frac{3}{8\pi^2}\frac{\overline{m}_s^2(Q^2)}{Q^2}\left\{1 + \frac{17}{3}\frac{\overline{\alpha}_s(Q^2)}{\pi} + \frac{24}{21}\frac{\overline{m}_s^2(Q^2)}{Q^2}\left(\frac{\pi}{\overline{\alpha}_s(Q^2)} - \frac{31}{24}\right)\right. \\
&\quad + \frac{12}{21}\frac{\overline{m}_s^4(Q^2)}{Q^4}\left(\frac{2\pi}{\overline{\alpha}_s(Q^2)} + \frac{59}{12}\right) + \frac{2\pi}{3}\frac{< \alpha_s G^2 >}{Q^4} \\
&\quad \left. - \frac{8\pi^2}{3}\frac{< m_s\overline{s}s >}{Q^4}\left[1 + \frac{14}{3}\frac{\overline{\alpha}_s(Q^2)}{\pi} + 2\frac{< \overline{u}u >}{< \overline{s}s >}\left(1 + \frac{17}{3}\frac{\overline{\alpha}_s(Q^2)}{\pi}\right)\right]\right\}
\end{aligned} \tag{4.9}$$

$\xi(Q^2)$ obeys then the FESR (derived in [15])

$$\int_0^{s_0}\frac{ds}{s}\frac{1}{\pi}Im\psi(s)|_{RES} = \frac{3}{8\pi^2}\overline{m}_s^2(s_0)s_0\left[1 + R_1(s_0) + \frac{24}{21}\frac{\overline{m}_s^2(s_0)}{s_0}\left(\frac{\pi}{\overline{\alpha}_s(s_0)} - \frac{31}{24}\right)\right] \tag{4.10}$$

where

$$R_1(s_0) = \frac{\overline{\alpha}_s(s_0)}{\pi}\left[\frac{17}{3} + 2 - \frac{2}{\beta_1}\left(\gamma_2 - \gamma_1\frac{\beta_2}{\beta_1}\right) + \frac{2\gamma_1\beta_2}{\beta_1^2}\ln\ln\frac{s_0}{\Lambda^2}\right] \tag{4.11}$$

In the above, s_0 is the assymptotic freedom threshold, which is the square of the energy at which we assume that the resonance contribution to the spectral function becomes similar to the expansion calculable in perturbative QCD.The constants in the sum rule are given by

$$\beta_1 = -\frac{9}{2},\quad \beta_2 = -8,\quad \gamma_1 = 2\ \ \text{and}\gamma_2 = 7.5833 \tag{4.12}$$

So, for the FESR (4.10), if we know the resonance contribution to $1/\pi Im\psi(s)$, then for some value of s_0 we can calculate the strange quark mass. Obviously, if $\hat{m}_s$ depended strongly on s_0, then it could not be determined. One expects, however, that a good saturation of the hadronic integral in the FESR should lead to a wide region in s_0 where $\hat{m}_s$ does not change appreciably. (This is a stability or duality region.) Experience from other channels indicates that this is normally the case whenever experimental data up to reasonably high energies is used in the FESR.

There is also a Laplace transform sum rule that we can obtain. The scalar correlator $\psi(q^2)$ defined above obeys the dispersion relation

$$\psi(Q^2) = \frac{1}{\pi}\int_0^\infty \frac{Im\psi(s)}{(s+Q^2)}ds, \tag{4.13}$$

valid up to two subtractions. Taking then two derivatives in order to get rid of these subtractions

$$\psi''(Q^2) = \frac{1}{\pi}\int_0^\infty \frac{Im\psi(s)}{(s+Q^2)^3}ds. \tag{4.14}$$

Referring again to the result in [3] we obtain

$$\begin{aligned}
\psi''(Q^2) \quad = \quad & \frac{3}{8\pi^2}\frac{\overline{m}_s^2(Q^2)}{Q^2}\left\{1+\frac{11}{3}\frac{\overline{\alpha}_s(Q^2)}{\pi} - 2\frac{\overline{m}_s^2(Q^2)}{Q^2}\left(1+\frac{28}{3}\frac{\overline{\alpha}_s(Q^2)}{\pi}\right)\right. \\
& -\frac{8}{7}\frac{\overline{m}_s^4(Q^2)}{Q^4}\left(\frac{\pi}{\overline{\alpha}_s(Q^2)}+\frac{149}{24}\right)+\frac{2\pi}{3}\frac{<\alpha_s G^2>}{Q^4} \\
& \left. +8\pi^2\frac{<m_s\bar{s}s>}{Q^4}\left[1+\frac{22}{3}\frac{\overline{\alpha}_s(Q^2)}{\pi}\right]\right\}
\end{aligned} \tag{4.15}$$

Applying the Laplace transform to improve this sum rule, we get

$$\hat{L}\left[\psi''(Q^2)\right] = \frac{1}{M^6}\int_0^\infty dse^{-s/M^2}\frac{1}{\pi}Im\psi(s) \tag{4.16}$$

where

$$\begin{aligned}
\hat{L}[\psi''(Q^2)] \quad = \quad & \frac{3}{8\pi^2}\frac{\hat{m}_s^2}{M^2}\frac{1}{(\frac{1}{2}\ln M^2/\Lambda^2)^{-2\gamma_1/\beta_1}} \\
& \cdot\left\{1+\frac{\overline{\alpha}_s(M^2)}{\pi}\left[\frac{11}{3}-\gamma_1\psi(1)+4\frac{\beta_2}{\beta_1^2}\ln\ln\frac{M^2}{\Lambda^2}-\frac{4}{\beta_1\gamma_1}\left(\gamma_2-\gamma_1\frac{\beta_2}{\beta_1}\right)\right]\right. \\
& -2\frac{\hat{m}_s^2}{M^2}\frac{1}{(\frac{1}{2}\ln M^2/\Lambda^2)^{-2\gamma_1/\beta_1}}\left[1+\frac{\overline{\alpha}_s(M^2)}{\pi}\left(\frac{28}{3}-2\gamma_1\psi(2)\right.\right. \\
& \left.\left. +8\frac{\beta_2}{\beta_1^2}\ln\ln\frac{M^2}{\Lambda^2}-\frac{8}{\beta_1\gamma_1}\left(\gamma_2-\gamma_1\frac{\beta_2}{\beta_1}\right)\right)\right] \\
& -\frac{4}{7}\frac{\hat{m}_s^4}{M^4}\frac{1}{(\frac{1}{2}\ln M^2/\Lambda^2)^{-4\gamma_1/\beta_1}}\left[\frac{149}{24}-\frac{\beta_1}{2}\ln\frac{M^2}{\Lambda^2}-\frac{\beta_1}{2}\left(\frac{6\gamma_1}{\beta_1}+1\right)\psi(3)\right. \\
& \left. +\left(12\frac{\beta_2}{\beta_1^2}+\frac{\beta_2}{\beta_1}\right)\ln\ln\frac{M^2}{\Lambda^2}-\frac{12}{\beta_1\gamma_1}\left(\gamma_2-\gamma_1\frac{\beta_2}{\beta_1}\right)\right] \\
& +\frac{\pi}{3}\frac{<\alpha_s G^2>}{M^4}+4\pi^2\frac{<m_s\bar{s}s>}{M^4}\left[1+\frac{\overline{\alpha}_s}{\pi}\left(\frac{22}{3}-\gamma_1\psi(3)\right.\right. \\
& \left.\left.\left. +4\frac{\beta_2}{\beta_1^2}\ln\ln\frac{M^2}{\Lambda^2}-\frac{4}{\beta_1\gamma_1}\left(\gamma_2-\gamma_1\frac{\beta_2}{\beta_1}\right)\right)\right]\right\}
\end{aligned} \tag{4.17}$$

Here $\psi(1) = -\gamma_E = -0.5772$ and $\psi(2), \psi(3), \ldots$ are defined by the recursive relation for the digamma function: $\psi(z+1) = \psi(z) + 1/z$. In this equation we have assumed that

$<\bar{s}s> / <\bar{u}u> = 1$; it turns out from the calculations that even if there are sizeable departures from unity, the results remain essentially unchanged.

We discuss now the hadronic spectral function. As stated in the previous chapter, this can be written as follows:

$$\frac{1}{\pi}Im\Pi(s)|_{HAD} = \frac{1}{\pi}Im\Pi(s)|_{RES}\theta(s_0 - s) + \frac{1}{\pi}Im\Pi(s)|_{AF}\theta(s - s_0) \tag{4.18}$$

The resonance contribution we infer from experiment, while the contribution from energies greater than s_0, the assymptotic freedom threshhold, we obtain from the QCD assymptotic freedom expression. This is given by [17]

$$\frac{1}{\pi}Im\psi(s)|_{AF} = \frac{3}{8\pi^2}\overline{m}_s^2 s\left(1 + \frac{11}{3}\frac{\overline{\alpha}_s}{\pi} + \ldots\right). \tag{4.19}$$

We need to know the resonance contribution for both the FESR and the Laplace type sum rules.

Although in principle we could fit the experimental data on the $K\pi$ channel with a purely algebraic form, we wish to discuss a physically motivated parametrization of the hadronic spectral function which in the end does the same job.

The definition of the hadronic spectral function is [18]

$$\frac{1}{\pi}Im\psi = \frac{1}{\pi}\frac{(2\pi)^4}{2}\sum_n |<0|\partial^\mu V_\mu|n>|^2\,\delta^4(q - p_n) \tag{4.20}$$

The summation should in principle extend over all intermediate states with the quantum numbers of $\partial^\mu V_\mu$, but in practice we take only the first few low-lying states. The appropriate intermediate states are $K^+\pi^0$ and $K^0\pi^+$. We then get

$$\begin{aligned}
\frac{1}{\pi}Im\psi &= \frac{(2\pi)^4}{2\pi}\int\!\!\int \frac{d\vec{q}_1}{2E_1}\frac{d\vec{q}_2}{2E_2}\frac{1}{(2\pi)^3}\frac{1}{(2\pi)^3}\delta^4(q - q_1 - q_2)\left[(d_{K^+\pi^0})^2 + (d_{K^0\pi^+})^2\right] \\
&= \frac{1}{2\pi}\frac{1}{(2\pi)^2}\frac{\pi}{2}\frac{1}{s}\sqrt{(s - s_+)(s - s_-)}\;\left[|d_{K^+\pi^0}|^2 + |d_{K^0\pi^+}|^2\right] \\
&= \frac{1}{16\pi^2}\sqrt{\left(1 - \frac{s^+}{s}\right)\left(1 - \frac{s^-}{s}\right)}\;\left[|d_{K^+\pi^0}|^2 + |d_{K^0\pi^+}|^2\right]\text{(see, e.g., [19])}
\end{aligned} \tag{4.21}$$

where

$$s_+ = \mu_K + \mu_\pi, \quad s_- = \mu_K - \mu_\pi, \tag{4.22}$$

and

$$d_{K^+\pi^0} \equiv \sqrt{2} < \pi^0|i\partial^\mu V_\mu|K^+>, \tag{4.23}$$

with a similar relation for $d_{K^0\pi^+}$.

Knowledge of these form factors can be gained from a fit to the data for $K^-p \to K^-\pi^+n$ scattering. Although this gives information for the $K^-\pi^+$ system, we can, using isospin relations, obtain the form factors for the $K^+\pi^0$ and $K^0\pi^+$ systems. We get

$$d_{K^0\pi^+} \equiv d \simeq 0.3GeV^2, \tag{4.24}$$

and, parametrizing the states according to isospin and using Clebsch-Gordan coefficients,

$$|d_{K^+\pi^0}| = \frac{1}{\sqrt{2}}d. \tag{4.25}$$

Hence

$$\begin{aligned}
\frac{1}{\pi} Im\psi &= \frac{1}{16\pi^2}\sqrt{\left(1-\frac{s^+}{s}\right)\left(1-\frac{s^-}{s}\right)}\,|d|^2\,(1+\frac{1}{2}) \\
&= \frac{3}{32\pi^2}\,|d|^2\sqrt{\left(1-\frac{s^+}{s}\right)\left(1-\frac{s^-}{s}\right)} \qquad (4.26)
\end{aligned}$$

This is not, however, the full picture. The $K\pi$ states can resonate to form the states $K_0^*(1430)$ and $K_0^*(1950)$ and thus we must impose a Breit-Wigner form onto the spectral function for each resonance. To simplify matters we choose some critical value s_c for the centre of mass energy squared, below which we deal with the Breit-Wigner form arising from the 1430 resonance, and above which we deal with that arising from the 1950. We choose s_c to be the point where the contribution from the first resonance is equal to that from the second. The value turns out to be $s_c = 2.75GeV^2$.Thus

$$\frac{1}{\pi} Im\psi(s) = \frac{3}{32\pi^2}\,|d|^2\sqrt{\left(1-\frac{s^+}{s}\right)\left(1-\frac{s^-}{s}\right)}\,\{BW_1(s)\theta(s_c-s)+BW_2(s)\theta(s-s_c)\}\,,$$

where

$$BW_1(s) = \frac{\left(M^2_{K^*(1430)}-s_+^{K\pi}\right)^2+M^2_{K^*(1430)}\Gamma^2_{K^*(1430)}}{\left(M^2_{K^*(1430)}-s\right)^2+M^2_{K^*(1430)}\Gamma^2_{K^*(1430)}} \qquad (4.27)$$

and similarly for $BW_2(s)$.

Furthermore, the $K^*(1950)$ can also couple to the $K^+\eta'$ intermediate state, so we must add a term to account for this. We can work out, using SU(3), that

$$\frac{< K^+|K^+\eta' >}{< K^+|K^0\pi^+ >} = -0.87\cdot\sqrt{2} \qquad (4.28)$$

and hence

$$\begin{aligned}
\frac{1}{\pi} Im\psi^{K\eta'} &= \theta(s-s_c)\frac{1}{16\pi^2}\left|0.87\cdot\sqrt{2}\,d\right|^2\sqrt{\left(1-\frac{s_+^{K\eta'}}{s}\right)\left(1-\frac{s_-^{K\eta'}}{s}\right)}\,BW_2(s) \\
&= \theta(s-s_c)\frac{3}{32\pi^2}\left|\frac{2}{3}\cdot 0.87\cdot\sqrt{2}\,d\right|^2\sqrt{\left(1-\frac{s_+^{K\eta'}}{s}\right)\left(1-\frac{s_-^{K\eta'}}{s}\right)}\,BW_2(s) (4.29)
\end{aligned}$$

Numerically, $|\frac{2}{3}\cdot 0.87\cdot\sqrt{2}\,d|^2 = 1.01 \simeq 1$.

So finally we get

$$\begin{aligned}
\frac{1}{\pi} Im\psi(s) &= \frac{3}{32\pi^2}\,|d|^2\left[\sqrt{\left(1-\frac{s_+^{K\pi}}{s}\right)\left(1-\frac{s_-^{K\pi}}{s}\right)}\,\{BW_1(s)\theta(s_c-s)+BW_2(s)\theta(s-s_c)\}\right. \\
&\left.+\sqrt{\left(1-\frac{s_+^{K\eta'}}{s}\right)\left(1-\frac{s_-^{K\eta'}}{s}\right)}\,BW_2(s)\,\theta(s-s_c)\right] \qquad (4.30)
\end{aligned}$$

For the sum rules we need to integrate this from zero to infinity. We must in fact begin our integration at the physical threshold $S_+ = (\mu_K + \mu_\pi)^2$. So

$$\int_0^\infty \frac{1}{\pi} Im\psi(s) = \int_{s_+}^{s_0} \frac{1}{\pi} Im\psi(s)|_{RES} + \int_{s_0}^\infty \frac{1}{\pi} Im\psi(s)|_{AF}. \tag{4.31}$$

In the next chapter, these integrals are evaluated, and the mass of the strange quark is then calculated from the sum rules.

Chapter 5

Results

To recap then, we have the sum rules :

$$\int_{s_+}^{s_0} \frac{ds}{s}\frac{1}{\pi} Im\psi(s)|_{RES} = \frac{3}{8\pi^2}\overline{m}_s^2(s_0)s_0\left[1 + R_1(s_0) + \frac{24}{21}\frac{\overline{m}_s^2(s_0)}{s_0}\left(\frac{\pi}{\overline{\alpha}_s(s_0)} - \frac{31}{24}\right)\right], \tag{4.10}$$

and

$$\frac{1}{M^6}\int_0^\infty dse^{-s/M^2}\frac{1}{\pi}Im\psi(s) = \hat{L}\left[\psi''\left(Q^2\right)\right], \tag{4.16}$$

which we need to solve for $\hat{m}_s$.

We first need to perform the integrals on the left hand side. For both of the sum rules we encounter an integral from s_+ to s_0 of $1/\pi Im\psi(s)|_{RES}$, multiplied by some function. We can then break up the integral from s_+ to s_0 into an integral from s_+ to s_c and an integral from s_c to s_0. For the Laplace sum rule, we also need to evaluate the integral of the spectral function multiplied by some function of s and M^2, from s_0 to infinity. It was found that, because of the exponential term, we could integrate up to one hundred instead of infinity, with no loss of accuracy. These integrals were all evaluated using a gaussian numerical integration routine, since none of them was analytically solvable.

For the FESR, we performed the integration for a range of s_0, starting at $s_0 = 2.0$ and moving up to $s_0 = 6.0$ in increments of 0.1. For the Laplace sum rule, we performed the integrations for $s_0 = 3.0,\ 3.5, \ldots, 7.5$ and for $M^2 = 1.0,\ 1.25, \ldots, 9.0$.

Then for a particular value of s_0 and, for the Laplace sum rule, for a particular value of M^2, we could solve the sum rules for $\hat{m}_s$. This was done as follows: The right hand side of each sum rule may be written as a polynomial in $\hat{m}_s$. Then using Newton's method, we solve the polynomial for $\hat{m}_s$ by a series of approximations. Newton's method says that, given some initial approximation x_1 to the root of a function $f(x)$, the next approximation to the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

So using this method, and using a seed value of $\hat{m}_s = 0.25\ GeV$, we obtained values of $\hat{m}_s$ for every value of s_0 (and M^2 in the case of the Laplace sum rule).

As a check, we notice that the Laplace sum rule is in fact a quadratic in $\hat{m}_s$, so instead of solving using Newton's method, we simply solved the quadratic for $\hat{m}_s$. We also used one of the CERN routines for the roots of a function. In both cases, the results agreed completely with our original method.

In all calculations, we used the values

$$\frac{\pi}{3} < \alpha_s G^2 >= 0.1\ GeV^4$$

and

$$< m_s \bar{s}s >= -0.0016\ GeV^4.$$

Furthermore, the calculations were all performed using $\Lambda = 100\ MeV$, and then repeated using $\Lambda = 200\ MeV$.

Using the results of these calculations, we can then, in the case of the FESR, plot graphs of $\hat{m}_s$ versus s_0, and, in the case of the Laplace sum rule, plot graphs of $\hat{m}_s$ versus M^2. Now, the crux of applying the sum rules is this: if we can, for reasonable values of s_0 and M^2, find a stable region on the graph (in which the mass $\hat{m}_s$ does not change appreciably over a wide range of s_0 or M^2), then we can find the mass of the strange quark. This mass is just the mass of $\hat{m}_s$ in the stable region.

In figure 5.1, graphs are plotted, using the FESR (4.10), of $\hat{m}_s$ versus s_0 for both $\Lambda = 100\ MeV$ and $\Lambda = 200\ MeV$. One can see that for both graphs, there is a wide region of stability from $s_0 \simeq 3.0\ GeV^2$ to $s_0 \simeq 6.0\ GeV^2$. In the stable region, we read

$$\begin{aligned} \hat{m}_s &= 284 - 288\ MeV \quad (\Lambda = 100\ MeV) \\ \hat{m}_s &= 238 - 245\ MeV \quad (\Lambda = 200\ MeV) \end{aligned} \tag{5.1}$$

Thus

$$\begin{aligned} \overline{m}_s(1GeV) &= 190 - 193\ MeV \quad (\Lambda = 100\ MeV) \\ \overline{m}_s(1GeV) &= 192 - 198\ MeV \quad (\Lambda = 200\ MeV) \end{aligned} \tag{5.2}$$

Here $\overline{m}_s$ is the running mass according to (2.29) evaluated at 1 GeV.

In figures 5.2 and 5.3, we have used the sum rule (4.16) to plot $\hat{m}_s$ versus M^2 taking first $\Lambda = 100\ MeV$ and then $\Lambda = 200\ MeV$, and for $s_0 = 5.5,\ 6.0,\ \ldots\ 8.0\ GeV^2$. We can see that the $\Lambda = 100\ MeV$ graph shows stability at reasonable M^2 for $s_0 = 6.0$, 6.5 and 7.0 GeV^2, while for $\Lambda = 200\ MeV$, the graph shows this stability at $s_0 = 6.0$ and 6.5 GeV^2. The dependence on both s_0 and M^2 is slight. Thus we read

$$\begin{aligned} \hat{m}_s &= 289 - 295\ MeV \quad (\Lambda = 100\ MeV) \\ \hat{m}_s &= 237 - 244\ MeV \quad (\Lambda = 200\ MeV) \end{aligned} \tag{5.3}$$

corresponding to $s_0 = 6 - 6.5\ GeV^2$. Hence

$$\begin{aligned} \overline{m}_s(1\ GeV) &= 193 - 198\ MeV \quad (\Lambda = 100\ MeV) \\ \overline{m}_s(1\ GeV) &= 191 - 197\ MeV \quad (\Lambda = 200\ MeV). \end{aligned} \tag{5.4}$$

Combining the results (5.1), (5.2), (5.3), (5.4), we may obtain a prediction for $\hat{m}_s$:

$$\hat{m}_s = 266 \pm 29\ MeV \tag{5.5}$$

and for the running mass

$$\overline{m}_s(1\ GeV) = 194 \pm 4\ MeV. \tag{5.6}$$

These values are consistent with the values from [2] of $\overline{m}_s(1\ GeV) = 192 \pm 15\ MeV$ and the value from current algebra of $\hat{m}_s = 288 \pm 48\ MeV$. However, because of the improved QCD expansion from [3] and the more complete phenomenological input (using data from [4]), our uncertainty is reduced.

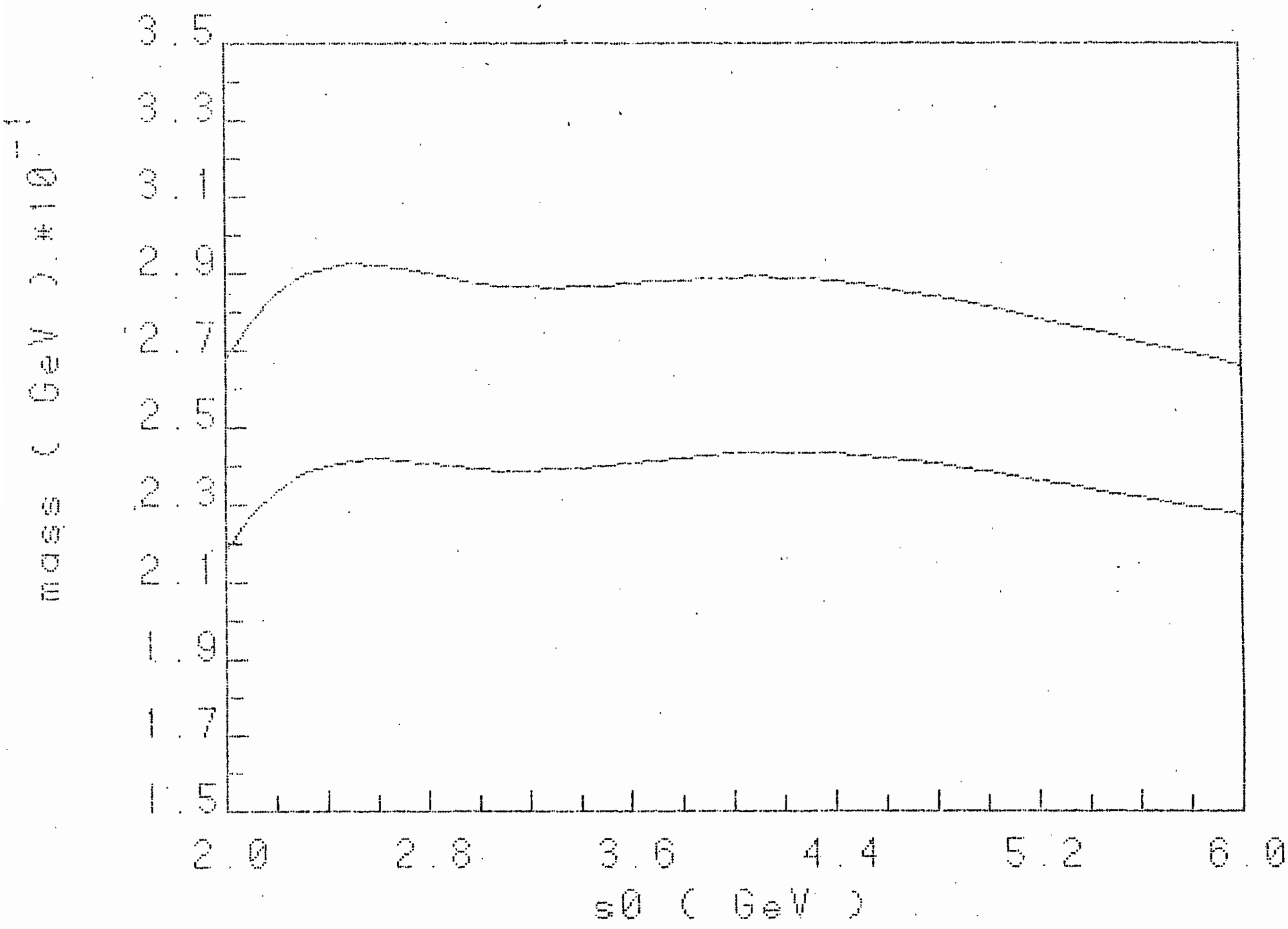

Figure 5.1: Graphs of $\hat{m}_s$ vs. s_0 for $\Lambda = 100\ MeV$ (top curve) and $\Lambda = 200\ MeV$ (bottom curve)

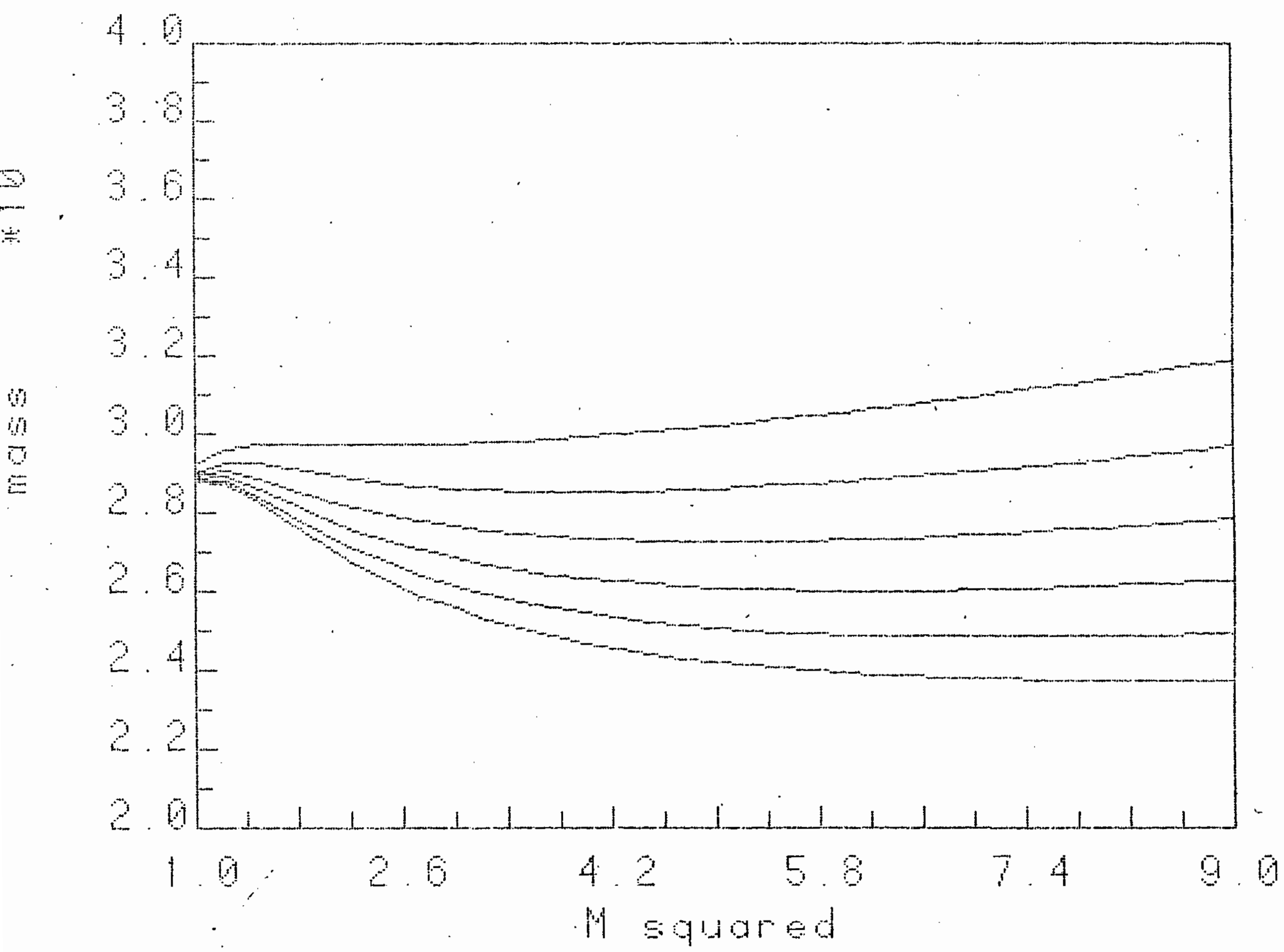

Figure 5.2: Graph of $\hat{m}_s$ vs. M^2, using the Laplace sum rule, for $\Lambda = 100\ MeV$ and $s_0 = 5.5,\ 6.0,\ ,\ \ldots\ ,\ 8.0$ (top to bottom)

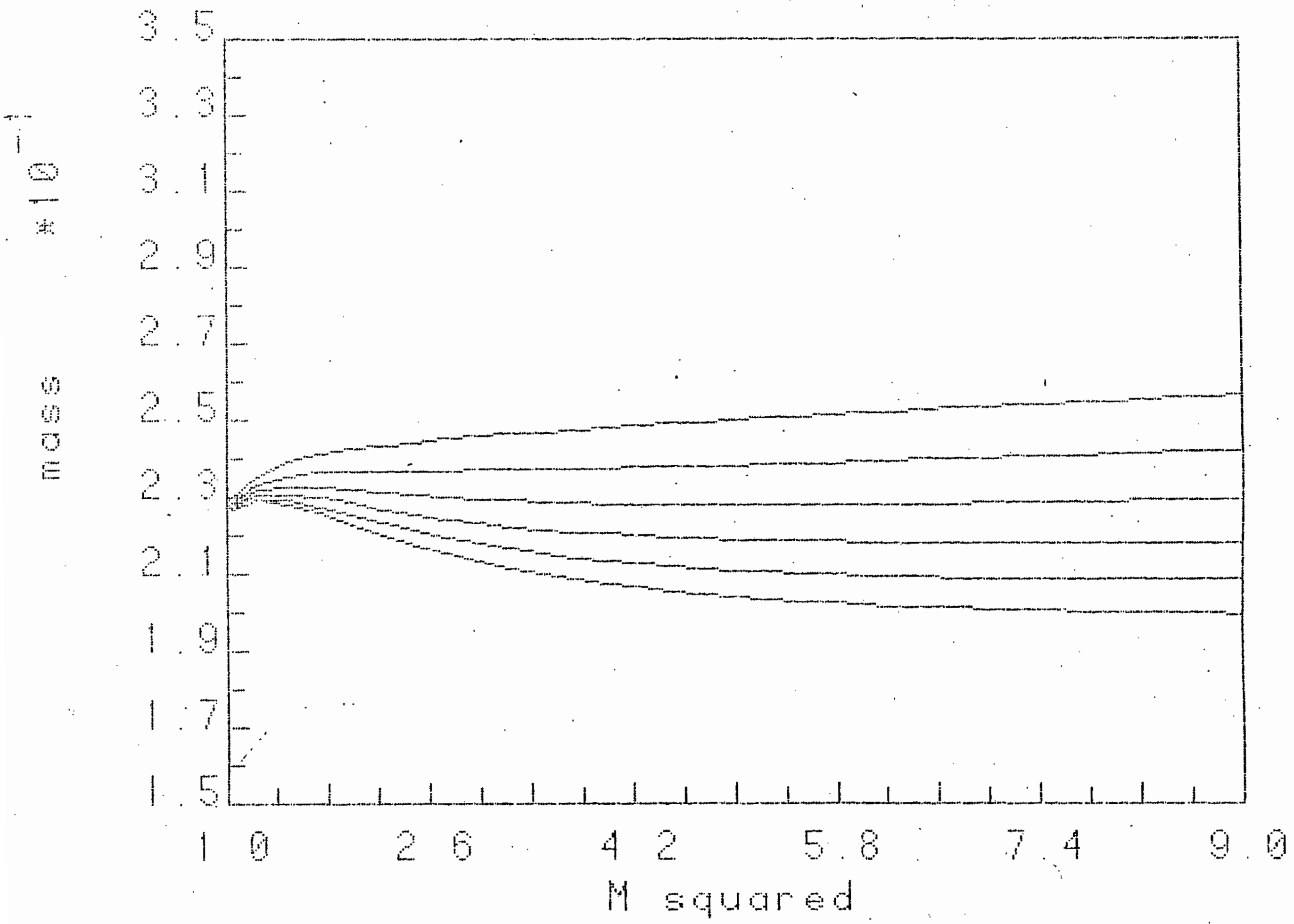

Figure 5.3: Graph of $\hat{m}_s$ vs. M^2, using the Laplace sum rule, for $\Lambda = 200\ MeV$ and$s_0 = 5.5,\ 6.0,\ ,\ \ldots\ ,\ 8.0$ (top to bottom)

Appendix A

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